Ballistic flights and random diffusion as building blocks for Hamiltonian kinetics

S. Denisov, J. Klafter, and M. Urbakh
School of Chemistry, Tel-Aviv University, Tel-Aviv 69978, Israel
(Received 29 May 2002; published 24 October 2002)

We propose a kinetic approach to transport in Hamiltonian systems with a mixed phase space. The approach is based on the decomposition of the dynamical picture into two contributions: (a) ballistic flights, and (b) random diffusion. The kinetic scheme leads to a stochastic process with statistical properties which are similar to those produced by the original Hamiltonian. We show that our approach helps in obtaining an insight into several properties of Hamiltonian kinetics such as anomalous diffusion, chaos-assisted population exchange, and current rectification. In particular, the chaos-assisted exchange offers a classical counterpart for the recently reported chaos-assisted tunneling.

DOI: 10.1103/PhysRevE.66.046217 PACS number(s): 05.45.Gg

I. INTRODUCTION

The issue of how statistical laws emerge from microscopic dynamical evolution has been a subject of interest for a long time [1]. In particular, the question of deterministic diffusion in Hamiltonian systems has been central in the field of nonlinear dynamics. Kinetic studies of Hamiltonian systems have been shown to provide a link between stochastic processes and deterministic dynamics [2] and have led to interesting examples of Brownian and anomalous motion [3].

In general, characteristic to Hamiltonian systems are mixed phase spaces, which consist of chaotic and regular regions. The presence of complex boundaries between these regions makes a complete separation into chaotic and regular regions impossible. Due to sticky barriers (formed by *cantori* [4]), a trajectory can be trapped for a long time near the corresponding regular islands and produce either long unidirectional flights or localized rotational motions. It is known by now that such long correlated motions cause the failure of the simple Brownian description of Hamiltonian kinetics [2,3]. Examples for non-Brownian behavior are, among others, the appearance of anomalous diffusion, for which the degree of anomaly depends on the statistical properties of the flights [2,3], strongly nonuniform mixing [5], and fractal conductance fluctuations in billiard transmitters [6]. Recently, it has been also shown that the generation of directed currents in Hamiltonian ratchets is obtained by introducing asymmetry into flights in opposite directions through breaking symmetry in the regular island structures in phase space [7,8].

In this paper we show how the Hamiltonian kinetics can be modeled using two types of ‘‘building blocks’’—ballistic flights (BF) and random diffusion (RD). Following the continuous-time random walk (CTRW) formalism [3], we construct a stochastic model which describes the transport properties of the original Hamiltonian in terms of these building blocks. We demonstrate that such an approach leads to new insights into phenomena typical of Hamiltonian systems with a mixed phase space.

The paper is organized as follows. In Sec. II we formulate the algorithm of the kinetic modeling using the BF and RD building blocks. For this purpose we need information about the statistical properties of the original Hamiltonian system. The result of such modeling is a kinetic scheme which leads to a process with the same basic statistical properties as of the trajectories in the Hamiltonian system. In Sec. III we show how this approach can be used for the description of three different effects: deterministic Hamiltonian diffusion, chaos-assisted population exchange between islands, and current rectification in Hamiltonian ratchets. We end with conclusions in Sec. IV.

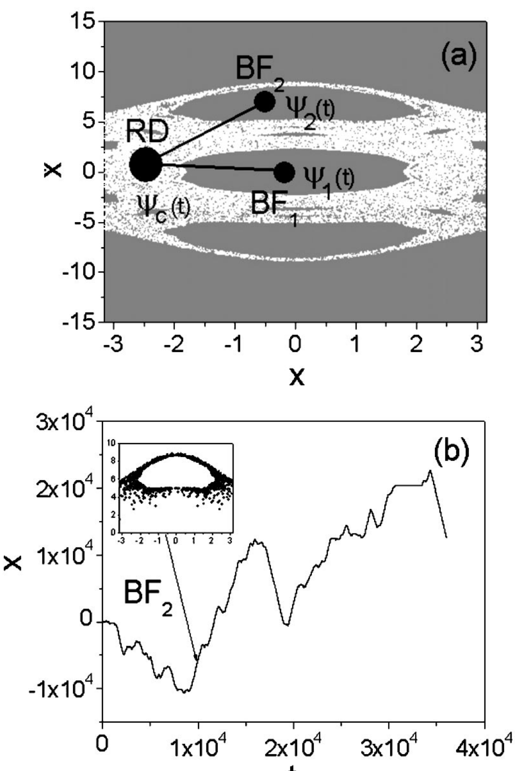

FIG. 1. (a) The kinetic scheme of a mixed phase space. Notations are given in the text. (b) A trajectory corresponding to the Hamiltonian in Eq. (4) with $\omega=2\pi$, $E=5$. The inset shows part of Poincarè section that corresponds to the long flight in the positive direction. All plotted quantities are dimensionless.

1063-651X/2002/66(4)/046217(6)/$20.00

II. BUILDING OF THE KINETIC SCHEME

Many Hamiltonian systems are generically nonintegrable and have a mixed phase space which contains chaotic regions and regular islands [9]. These islands are impermeable to chaotic trajectories and, at first glance, do not contribute to the system kinetics. But the exclusion of the regular islands from consideration is incorrect due to the presence of "sticky" complex boundaries between chaotic and regular regions. Due to a set of barriers, formed by cantori [4] a trajectory can be trapped for a long time near the corresponding islands. This leads to the appearance of long ballistic flights (in the case of nonzero winding numbers v of the islands) or to a localized rotational motion (for the winding number $v=0$). For some islands the sticking times near the boundaries can be anomalously long resulting in Lévy walks [2,3]. So, the regular islands manifest themselves *dynamically* as segments of ballistic motion in the trajectories [see inset in Fig. 1(b)]. The trajectories between two consecutive flights display random Brownian motion that can be described by some diffusion coefficient.

Following the CTRW formalism [3], the Hamiltonian kinetics can be represented as a set of alternating pieces of random diffusion (RD) and ballistic flights (BF). The applicability of the CTRW description stems from the assumption that the presence of a random diffusion with a fast decay of correlations leads to weak correlations between consecutive flights (since they are always separated by long enough random motion in chaotic area). From the point of view of kinetics, ballistic flights can be characterized by the value of the velocity and the duration of a single flight [3]. The velocity of a flight is given by the winding number v of the corresponding island. The duration of the flights can be completely characterized by the probability density function (PDF) of the sticking times to the boundary of the islands [3].

Assume that there exist N different islands with winding number v_i, $i=1,\ldots,N$, which are embedded in the chaotic area. Every island is characterized by a PDF $\psi_i(t)$ of sticking times. The random diffusion is characterized by a PDF $\psi_c(t)$. The probability of sticking to the ith island, after the end of a random diffusion event, is p_i. The probability p_i is the weight factor of island i, namely, its contribution to the total dynamical evolution of a system. Due to the conservation of probability $\Sigma_{i=1}^{n} p_i=1$.

The kinetics of the system can be obtained from the propagator $P(x,t)$, i.e., the probability density of a particle to be at x at time t [4]. This function can be constructed from the elementary events which include flights and random diffusion. We define a flight event by the probability density [3]

$$\Psi_i(x,t)=p_i\delta(x-v_i t)\psi_i(t), \tag{1}$$

which accounts for the velocity v_i and the distribution of sticking times $\psi_i(t)$, which correspond to island i.

The random diffusion events are chosen from the PDF,

$$\Psi_c(x,t)=\frac{1}{\sqrt{\pi D t}}e^{-x^2/Dt}\psi_c(t). \tag{2}$$

The propagator for time $t=MT$ (where T is some natural system time scale, i.e., the period of an external drive) is the convolution of all the elementary PDF's,

$$P(x,MT)=Q\sum_{n=1}^{M}\sum_{k=1}^{M-n}\cdots\sum_{l=1}^{M-n-\cdots-g}\Psi_c[x,(M-n-\cdots-l)T]\Psi_1(x,nT)\cdots\Psi_N(x,lT), \tag{3}$$

where the number of sums equals the number of relevant islands N, and Q is a normalization constant.

All the needed information about the system kinetics is contained in the PDF's of the flight times, the times of random diffusion, and the island weight factors p_i. This requires a preliminary examination of the phase space structure. In the following section we describe a numerical procedure which helps to obtain the parameters of the kinetic scheme.

The kinetic model can be viewed as the children's construction game LEGO: several BF-blocks, each with its own "color" [PDF $\psi_i(t)$], which are attached to the RD blocks [with a "color" PDF $\psi_c(t)$]. Every attachment is characterized by a "strength" p_i. The evolution of the original Hamiltonian system can be modeled as walking over the corresponding kinetic scheme [see Fig. 1(a)]. A particle jumps randomly (following the probabilities p_i) to the ith BF block and spends there some time t_i chosen from the PDF $\psi_i(t)$. This stage corresponds to the ballistic flight with velocity v_i and duration t_i. Then it jumps back to the RD block and performs a random motion with a diffusion coefficient D and duration t_c chosen from the PDF ψ_c. This process is then iterated again and again. Each block is depicted in the system as a vortex.

The functions $\psi_i(t)$ and $\psi_c(t)$ have finite first moments, due to the Kac theorem about the finiteness of recurrence times in a Hamiltonian system [10]. Thus the mean times, $\langle t_i\rangle$ and $\langle t_c\rangle$, which the particle spends in every block are finite.

III. APPLICATIONS

In this section we demonstrate how the above proposed kinetic approach can be used for the description and understanding of several phenomena of Hamiltonian kinetics.

As an example we consider the following Hamiltonian, which describes the classical motion of a particle in a spatially standing wave with a modulating amplitude $E\cos^2(\omega t)$ [9],

$$H(p,x,t)=\frac{p^2}{2}+E\cos(x)\cos^2(\omega t). \tag{4}$$

The Hamiltonian is time and space periodic with periods $T=\pi/\omega$ and $L=2\pi$, correspondingly. Such a Hamiltonian system has been realized in atomic optics experiments, probing the motion of atoms in a wave produced by a laser field [11].

We first study the case of chaotic diffusion in the system described by Eq. (4), using our building-blocks approach. Then we investigate the effect of chaos-assisted transport between two regular islands in the phase space of $H(p,x,t)$ in Eq. (4). Finally, we study a current rectification mechanism in a Hamiltonian ratchet system, which is created by switching on an additional standing wave in the system in Eq. (4).

(i) *Chaotic diffusion*. The Hamiltonian in Eq. (4) is symmetric with respect to time and space reversal transformation $\{t\to -t, x\to -x\}$, so a particle, whose dynamics obeys Eq. (4), performs a diffusive motion with a zero drift. The symmetry conditions mean that all islands with nonzero winding number appear in phase space in pairs, such that $\{\psi_i(t)=\psi_{-i}(t),\ p_i=p_{-i},\ v_i=-v_{-i}\}$. The corresponding kinetic scheme has therefore symmetric blocks, BF_i^+ and BF_i^-, with opposite velocities but equal "colors" $\psi(t)$ and "strengths" p.

In Fig. 2(a) we show the propagator $P(x,t)$ for a fixed time $t=100T$, which has been obtained by averaging over 10^5 trajectories. The peaks in the propagator correspond to ballistic flights which a particle performs when it sticks to an island. The location of each peak is determined by the corresponding winding number. One can use the propagator for a given time to identify all relevant flights. In our case there are only two main symmetry related regular islands, R_+, R_-, which lie at the border of the stochastic layer and have winding numbers $v_\pm=\pm 3L/2T$. The corresponding kinetic scheme has therefore three blocks which are: a RD block and two symmetry related BF blocks, BF_i^+ and BF_i^-.

In order to determine the PDF, $\psi_i(t)$, we use a velocity gated technique [8]. After each fixed time step $t_{step}=4T$ we check the displacement of the particle $\Delta x_{step}=x(t+t_{step})-x(t)$. If the resulting velocity $\Delta x_{step}/t_{step}$ is close to the corresponding winding number v (within an uncertainty of 5%), we define this as a flight. Using this technique it is possible to determine the the turning points and duration of single flight segments (with an uncertainty of $4T$), and, correspondingly, the duration of the random diffusion. We have found that the PDF $\psi(t)$ for the single flight duration follows an asymptotic power-law behavior $\psi(t)\sim t^{-\alpha}$ with the exponent $\alpha\simeq 2.35$. The BFs lead therefore to a Lévy walk process and the overall diffusion has a strong anomalous character. The chaotic lifetime PDF displays a well pronounced Poisson distribution $\psi_c(t)\simeq(1/\tau_c)e^{-t/\tau_c}$ with $\tau_c\approx 345$. It is clear that for long enough times of the system evolution the contribution of the random diffusional motion to the particle displacement is negligible. So one can assume that here $D=0$. This means that flights with a power law PDF $\psi(t)$ dominate the long time scale behavior, in particular, the mean squared displacement. For simulations we used the following PDF [12]:

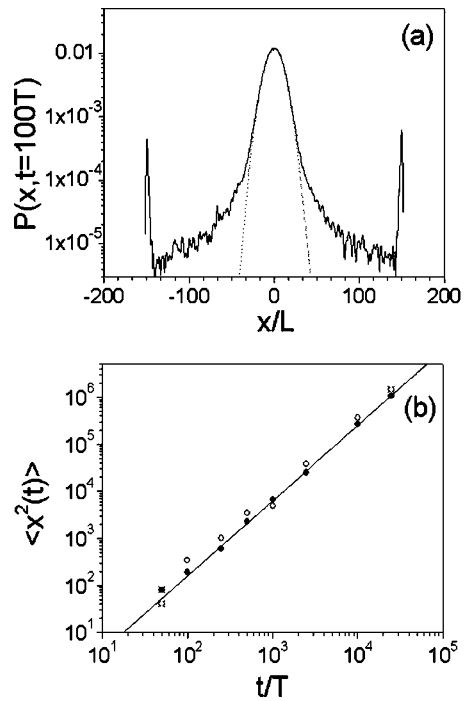

FIG. 2. (a) The propagator for a fixed time $t=100T$ for the Hamiltonian given by Eq. (4), $E=1$, $\omega=0.53$. Dotted curve shows the Gaussian approximation of a central part of the propagator. (b) Mean square displacement for the anomalous diffusion within chaotic layer averaged over 10^4 trajectories (open circles) and for the stochastic process described by the kinetic scheme. The straight line corresponds to the asymptotic $\langle x^2\rangle\propto t^\gamma$, $\gamma=1.6$. All plotted quantities are dimensionless.

$$\psi(t)=\begin{cases}0, & t<t_c\\ At^{-\alpha}, & t\geq t_c,\end{cases} \tag{5}$$

where A is a normalization constant, and $t_c=4T$. The results of the simulation are shown in Fig. 2(b). For long times we obtain anomalous enhanced diffusion which yields the mean square displacement

$$\langle x^2\rangle\propto t^\gamma \tag{6}$$

with $\gamma=1.6$. The latter value is very close to the value $4-\alpha=1.65$ given by the theory in Ref. [3]. The evolution of the mean square displacement obtained from the kinetic model for times $t>10^4T$ is close to that obtained from the real Hamiltonian kinetics [Fig. 2(b)].

(ii) *Population exchange between the islands of stability*. Let us now consider the situation where there are only two symmetry-related islands, R_- and R_+, embedded in the chaotic sea. A particle, initially trapped near one of these islands, moves away from its initial location and performs a random walk before it sticks again to an island. Chaotic diffusion can be viewed as some "communication channel" between the islands R_- and R_+. In the case of an ensemble of particles, initially prepared near one of the islands, this channel provides a possibility of population exchange between the islands. How would the particles redistribute due to this chaos-assisted "communication" between the islands?

Contrary to the previous diffusion problem, where for long time we assume $D=0$, here the random diffusion is essential to facilitate the exchange process. This classical problem is interesting in the context of the chaos-assisted tunneling effect, which has been observed recently in cold-atom experiments [13,14]. Here we look at the classical counterpart of the effect.

As a model we use the Hamiltonian in Eq. (4) with parameters taken from the experiment in Ref. [13]: $\omega=\pi, E=21$. The corresponding Poincarè section has two symmetry-related islands, R_+ and R_-, $v_\pm=\pm 2\pi$ [Fig. 3(a)] [15].

The corresponding kinetic scheme, as in the previous case, has three blocks, RD, BF_-, and BF_+. Here we are interested in relaxation dynamics from the initially prepared asymmetric state, where all the particles are located around R_+. For preparation of the initial ensemble we use our BF-RD building blocks scheme and a trajectory of *one* particle. Using the velocity gated technique we check the flight status of the particle. If the particle performs the flight with a duration of at least $2T$, we take the last coordinate, $[x(t);p(t)]$, as a point for the initial ensemble [see Fig. 3(a)]. Following the trajectory of only *one* particle, using the BF-RD approach, we extract information about the relaxation of initially "near-island" *ensemble*.

As in Ref. [13] we are interested in the average velocity

$$V(nT)=\frac{1}{S}\sum_{j=1}^{S}[x_j(nt)-x_j(nt-T)], \tag{7}$$

where S is number of particles in the ensemble. The evolution of the average velocity $V(nT)$ (for an ensemble with $S=10^4$ particles) is shown in Fig. 3(b).

The characteristic time for the population exchange mediated by chaotic diffusion corresponds to the first minimum in the time dependence of the mean ensemble velocity. This occurs at $t_{exch}\simeq 9T$. For times $t\sim t_{exch}$ a fraction of the particles accumulates near the opposite island R_-. Then this process is reiterated and a small fraction of particles reaccumulates near R_+ at time $2t_{exch}$ which corresponds to the local maximum in the velocity $V(nT)$.

The exchange of populations takes place on the background of a slow process during which particles "evaporate" from the vicinity of R_+ into the chaotic area [see inset in Fig. 3(b)]. This process is governed by the PDF of escape times with the asymptotic behavior $\psi_{esc}\sim t^{-\gamma_{esc}}$, where $\gamma_{esc}=\alpha-2$ [16]. For the parameters chosen here we obtain $\alpha=2.6$. Using the velocity gate technique (with an accuracy 3% and duration T) we obtain the PDF for the duration of random diffusion events between two consecutive flights in *opposite* directions [see Fig. 3(b)]. This PDF has a unique maximum near t_{exch}. We assume that this PDF that accounts for events between flights in *opposite* direction (exchange between BF^+ and BF^-) contains the information on t_{exch} which is observed experimentally. Interestingly, one faces here two processes that dominate the exchange on different times scales: the fast population exchange at early times and the slow "evaporation" at long times. We conclude that the clasically observed population exchange is mainly a short time effect.

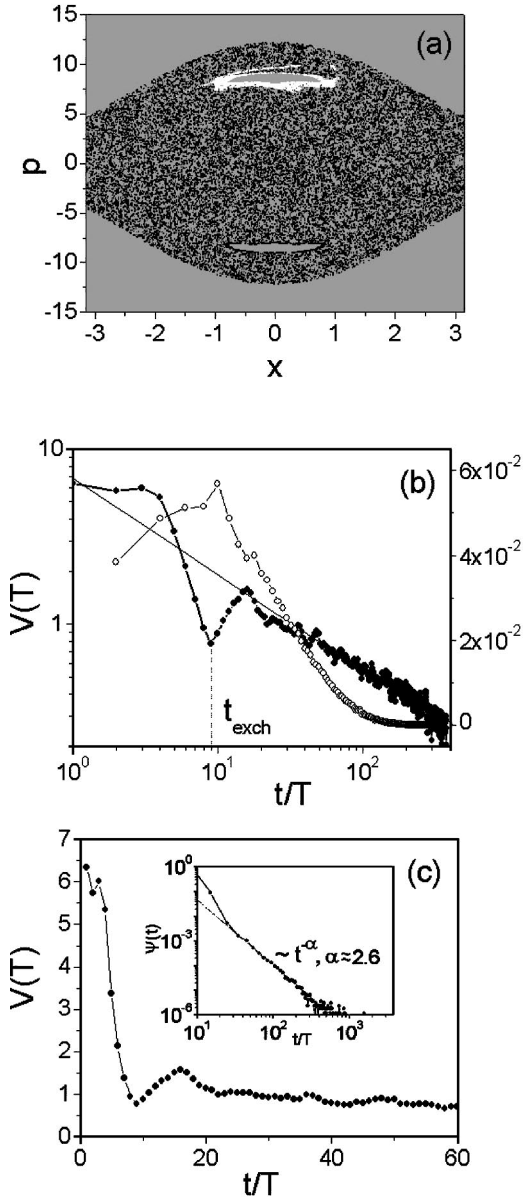

FIG. 3. (a) Poincarè section for the Hamiltonian in Eq. (4), $E=21$, $\omega=\pi$. White area shows the initial ensemble located around the island R_+. (b) Time dependence of the ensemble averaged velocity, Eq. (7)(filled circles). Initial conditions as in Fig. 3(a). The straight line corresponds to the asymptotic decay due to evaporation, $\psi(t)\sim t^{-0.6}$. The superimposed curve is the PDF for duration of random walk events between consecutive flights in opposite directions (open circles). (c) Time dependence of the ensemble averaged velocity for a short times scale. The inset shows the PDF for sticking times near a ballistic island. All plotted quantities are dimensionless.

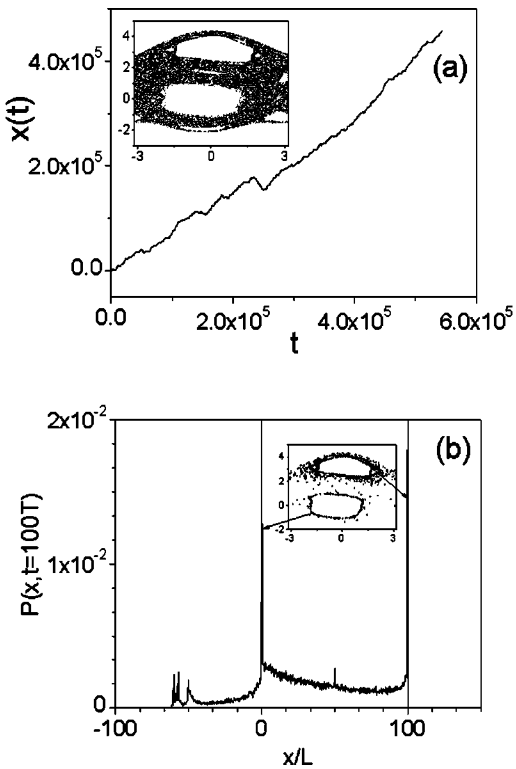

FIG. 4. (a) $x(t)$ vs t for the Hamiltonian ratchet, Eq. (8), $E=1$, $\omega=1.5$, $E_a=0.5$, $\phi=1.2$, $\tau=0.8$. Inset shows the corresponding Poincaré section. The trajectory starts in the chaotic sea. (b) The propagator for a fixed time $t=100T$ for the Hamiltonian ratchet, Eq. (8). The figure demonstrates the strong current due to asymmetric weight of positive flights. Inset displays sticky islands which correspond to the main peaks in the propagator. All plotted quantities are dimensionless.

Our kinetic scheme is close in essence to the three-state model of chaos assisted tunneling [17]; in particular, the value of the characteristic time for population exchange in the classical case (in units of the driving period T) is close to the time observed in real cold-atom systems [13].

(iii) *Hamiltonian ratchet*. Following the *ratchet* idea in Ref. [18], it is possible to obtain a dc current in a Hamiltonian system using zero-mean external driving [7,8,19]. In order to do this we must break the time/spatial reversal symmetry of the system [19]. All relevant symmetries of the Hamiltonian in Eq. (4) can be broken by switching on an additional standing wave, shifted, in the time and space with respect to the first one,

$$H_a(p,x,t)=H(p,x,t)+E_a\cos(x+\phi)\cos^2(\omega t+\tau), \quad (8)$$

where $H(p,x,t)$ is the Hamiltonian in Eq. (4), and ϕ and τ are spatial and temporal shift constants and E_a is the amplitude of the second standing wave. While in cases (i) and (ii) we have obtained an unbiased diffusion here we achieve directionality.

Switching on the second term in Eq. (8), which provides symmetry breaking in the island structure within the chaotic layer, results in an asymmetric overlap of the main chaotic layer with the layer of ballistic islands [see Fig. 4(a)]. This overlap generates a current. Current inversion (mirroring the layers overlap) can be obtained by a simple shift inversion $\phi\to-\phi$ or $\tau\to T-\tau$.

It has been shown in Refs. [7,8], that a directed current stems from ballistic islands which are not compensated by symmetry-related islands with opposite winding numbers. Moreover, the chaotic diffusion does not contribute to the current rectification [8]. Within the CTRW formalism, using the asymmetric flight processes approach [20], it is easy to obtain an expression for the current in the terms of BF and RD blocks [7,8],

$$J=\frac{\sum_{i=1}^{N} p_i v_i\langle t_i\rangle}{\sum_{i}^{N} p_i\langle t_i\rangle+\langle t_c\rangle}, \quad (9)$$

where $\langle t_i\rangle=\int_0^\infty t\psi_i(t)dt$ and $\langle t_c\rangle=\int_0^\infty t\psi_c(t)dt$.

In Fig. 4(b) we show the propagator for the Hamiltonian in Eq. (2) for a fixed time $t=100T$. One can see that the main contribution to the particle transport in the positive direction comes directly from the main ballistic island with $v=L/T=1.5$. The localized rotation is governed by the regular island R_0 with winding number $v=0$ near $(x,p)=(0,0)$. Namely, the corresponding kinetic scheme has three blocks—RD, BF_0, and BF_+. In this case, for the calculation of a current we need the mean time $\langle t_+\rangle$ of the flight near R_+, the mean time for localization $\langle t_0\rangle$ near R_0, and the mean time $\langle t_c\rangle$ of random diffusion [21]. For the numerically obtained values, $\langle t_+\rangle\approx163$, $\langle t_0\rangle\approx51$, $p_+\approx0.44$, and $\langle t_c\rangle\approx24$, expression (9) yields $J_{model}\approx0.87$, which is very close to the result $J_{num}\approx0.9$ of direct numerical integration of the Hamiltonian in Eq. (8) (see Fig. 4(a)).

IV. CONCLUSION

We have presented a BF-RD building blocks approach to kinetics of Hamiltonian systems with a mixed phase space. This approach provides a tool for understanding and analyzing different phenomena such as anomalous diffusion, population exchange between islands of stability, and current rectification in Hamiltonian ratchets. The BF-RD model reproduces nontrivial peculiarities of Hamiltonian kinetics and allows to obtain new characteristics of dynamics in mixed phase spaces, such as the characteristic time in chaos-assisted population exchange between islands.

Recently hierarchical graphs have been introduced in order to mimic the dynamics near islands with hierarchical structures [22]. These hierarchies lead to a power-law behavior of the sticking time PDF [22]. Such an approach might help obtain the PDFs discussed above from walking on hierarchical graphs [23]. We believe that the proposed kinetic scheme idea provides a different possibility for modeling of dynamics in mixed phase spaces, especially when combined with a PDF derivation. Our model allows also for further extensions such as the inclusion of correlations between sticking processes to different islands.

ACKNOWLEDGMENTS

The authors thank the support of INTAS project 00-0847 (J.K.) and of the Israel Science Foundation (M.U.).

[1] P. Ehrenfest and T. Ehrenfest, *The Conceptual Foundations of the Statistical Approach in Mechanics* (Cornell University Press, Ithaca, NY, 1959); N. S. Krylov, *Works on the Foundation of Statistical Physics* (Princeton University Press, Princeton, NJ, 1979).

[2] M.F. Shlesinger, G.M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).

[3] J. Klafter and G. Zumofen, Phys. Rev. E **49**, 4873 (1994); J. Klafter, G. Zumofen, and M.F. Shlesinger, Lect. Notes Phys. **450**, 196 (1994).

[4] R.S. MacKay, J.D. Meiss, and I.C. Percival, Physica D **13**, 55 (1983); J.D. Meiss and E. Ott, *ibid.* **20**, 397 (1986).

[5] S. Wiggins, *Chaotic Transport in Dynamical Systems* (Springer, New York, 1992).

[6] R. Ketzmerick, Phys. Rev. B **54**, 10 841 (1996); A.S. Sachrajda *et al.*, Phys. Rev. Lett. **80**, 1948 (1998).

[7] S. Denisov and S. Flach, Phys. Rev. E **64**, 056236 (2001).

[8] S. Denisov, J. Klafter, M. Urbakh, and S. Flach, Physica D **170**, 131 (2002).

[9] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics: From the Pendulum to Turbulence and Chaos* (Harwood Academic, New York, 1988); M. Glück, A.R. Kolovsky, and H.J. Korsch, Physica D **116**, 283 (1998).

[10] N. Kac, *Statistical Independence in Probability, Analysis, and Number Theory* (Mathematical Association of America, Oberlin, OH, 1959).

[11] F.L. Monte *et al.*, Phys. Rev. Lett. **75**, 4598 (1995); J.R. Robinson *et al.*, *ibid.* **76**, 3304 (1996); B.G. Klappauf, W.H. Oskay, D.A. Steck, and M.G. Raizen, *ibid.* **81**, 4044 (1998).

[12] For generation of random variable x with distribution (5) we have used the random variable ξ with uniform distribution in unit interval and transformation $x = t_c \xi^{-1/\alpha - 1}$.

[13] D.A. Steck, W.H. Oskay, and M. Raizen, Science **293**, 274 (2001).

[14] W.K. Hensinger *et al.*, Nature (London) **412**, 52 (2001).

[15] Some small islands are also present inside the chaotic area and at the borders of the chaotic layer, but their influence is negligible due to weak stickiness.

[16] G. Zumofen and J. Klafter, Phys. Rev. E **59**, 3756 (1999).

[17] D.A. Steck, W.H. Oskay, and M.G. Raizen, Phys. Rev. Lett. **88**, 120406 (2002).

[18] I. Goychuk and P. Hänggi, in *Stochastic Processes in Physics*, edited by J. Freund and T. Pöschel (Springer, Berlin, 2000), p. 7; P. Reimann, Phys. Rep. **361**, 57 (2002).

[19] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 2358 (2000).

[20] E.R. Weeks and H.L. Swinney, Phys. Rev. E **57**, 4915 (1998).

[21] The convergenge of the first moment $\langle t_i \rangle = \int_0^\infty t \psi_i(t) dt$ to the asymptotic value is much faster than that of the PDF $\psi_i(t)$. So, the direct calculation of the mean time $\langle t_i \rangle$ is more effective.

[22] L. Hufnagel, R. Ketzmerick, and M. Weiss, Europhys. Lett. **54**, 703 (2001).

[23] F. Barra and P. Gaspard, Phys. Rev. E **63**, 066215 (2001).